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On Solution of $AX + \alpha XB = A$, Range Inclusion and Parallel Sum*

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INTRODUCTION

The purpose of this paper is to describe recent reasearch on the range inclusion and parallel sum in order to find the solutions of the class of operator equations

$$\alpha XB + AX = A, \qquad \text{for all real } \alpha > 0.$$

Throughout this work, $\mathscr{H}$ will denote a complex Hilbert space, and A and B will be positive bounded linear operators in $\mathscr{H}$. Recall that $A \geqslant B$ means that $A - B$ is positive. The identity operator $\mathscr{H}$ will be denoted by I. We shall write $A + \lambda$ in place of $A + \lambda I$, where λ is a scalar. The range of the operator A is denoted by Range(A), the null space by Ker(A).

In Section 1 the definition of parallel sums of (possibly singular) positive bounded operators is given and the class UUP and range inclusion are discussed. In Section 2, we give a sufficient condition for the operator equation

$$\alpha XB + AX = A \qquad (\alpha > 0)$$

to have a positive solution. This class of operator equation arises in various practical situations, including the study of boundary value problems in partial differential equations, in sensitivity analysis of complex systems, and in the study of optimal control.

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1. The Parallel Sum and the Class UUP

The concept of a parallel sum has been studied by Anderson and Trapp [2], Fillmore and Williams [5], Green and Morley [6, 7], and Bunce [3].

If A and B denote positive operators and if $A+B$ is invertible in $B(\mathscr{H})$, the parallel sum $A : B$ is defined by

$$A : B = A(A+B)^{-1}B.$$

Note that if A and B are invertible in $B(\mathscr{H})$ then $A : B = (A^{-1}+B^{-1})^{-1}$. If $A+B$ is not invertible one can set

$$A : B = \lim(A+\varepsilon I) : (B+\varepsilon I).$$

The last limit exists when taken in the strong operator topology [2, 6]. The following characterization is due to Green and Morley.

Theorem 1.1 [6]. *The net $\{(A+\varepsilon) : (B+\varepsilon)\}_{\varepsilon>0}$ is norm convergent if and only if for every representation* Π *of the C^*-algebra $B(\mathscr{H})$ we have*

$$\Pi(A) : \Pi(B) = \Pi(A : B).$$

That is, this net is norm convergent if and only if parallel sum operation, applied to the (A, B) commutes with every representation of $B(\mathscr{H})$. In this case the pair (A, B) is said to be *uniformly and universally parallelizable* or of *class* UUP. This definition is due to Green and Morley, who studied this class [6].

Theorem 1.2 [6]. *Let $AB = BA$. Then (A, B) is of class* UUP.

The following results are due to the author and give a partial description of the class UUP.

Theorem 1.3 [9]. $\mathrm{Range}(A+B) \supseteq \mathrm{Range}(A)$ *if and only if* $\mathrm{Range}(A+B) \supseteq \mathrm{Range}(B)$.

Theorem 1.4 [8]. *If $AB+BA+B^2 \geqslant 0$, then* $\mathrm{Range}(A+\alpha B) \supseteq \mathrm{Range}(A)$, *for all $\alpha \geqslant 1$.*

Theorem 1.5 [8]. *If $AB+BA \geqslant 0$, then* $\mathrm{Range}(A+\alpha B) \supseteq \mathrm{Range}(A)$ *for all real $\alpha > 0$.*

Note that Theorem 1.5 implies Theorems 1.4 and 1.2.

Morley [7] and Bunce [3] have established the following result independently.

THEOREM 1.6. *If* $\mathrm{Range}(A+B) \supseteq \mathrm{Range}(A)$, *then* (A, B) *is of class* UUP.

We need the following result, which is of Theorem 1 of [4].

THEOREM 1.7. *Let A and B be bounded operators on the Hilbert space $\mathscr{H}$. The following statements are equivalent.*

(1) $\mathrm{Range}(B) \supseteq \mathrm{Range}(A)$.

(2) $AA^* \leqslant \lambda^2 BB^*$ *for some* $\lambda \geqslant 0$.

(3) *There exists a bounded operator C on $\mathscr{H}$ so that $A = BC$.*

Moreover, if (1), (2), *and* (3) *are valid, then there exists a unique operator C so that*

(a) $\|C\|^2 = \inf\{\mu \mid AA^* \leqslant \mu BB^*\}$,

(b) $\mathrm{Ker}(A) = \mathrm{Ker}(C)$,

(c) $\mathrm{Range}(B^*)^- \supseteq \mathrm{Range}(C)$.

The following theorem generalizes some of the work done by Green and Morley [6], Bunce [3], and the author [9].

THEOREM 1.8.[1] *Let A, B, and C be positive operators. If $AB + CA \geqslant 0$, then $(A, \alpha(B+C))$ is* UUP *for all $\alpha > 0$, and for a fixed $\alpha > 0$*

(1) $\mathrm{Range}[A + \alpha(B+C)] = \mathrm{Range}(A) + \mathrm{Range}(B+C)$;

(2) $\mathrm{Range}[A + \alpha(B+C)] \supseteq \mathrm{Range}(A)$;

(3) $\mathrm{Range}[A + \alpha(B+C)] \supseteq \mathrm{Range}(B+C)$;

(4) $\|A[A + \alpha(B+C) + \varepsilon]^{-1}\| \leqslant M_1$ *for some real M_1 and all* $\varepsilon > 0$;

(5) $\|(B+C)[A + \alpha(B+C) + \varepsilon]^{-1}\| \leqslant M_2$ *for some real M_2 and all* $\varepsilon > 0$.

Moreover, (1), (2), (3), (4), *and* (5) *are equivalent.*

Proof. If $AB + CA \geqslant 0$, then

$$AB + CA + BA + AC = A(B+C) + (B+C)A \geqslant 0.$$

It follows from Theorem 1.5 that

$$\mathrm{Range}[A + \alpha(B+C)] \supseteq \mathrm{Range}(A) \qquad \text{for all real } \alpha > 0.$$

[1] The proof of equivalence of (2) and (4) in a special case is due to J. Bunce [3]

Therefore $[A, \alpha(B+C)]$ is UUP for all real $\alpha > 0$. Now it suffices to show that (1) is equivalent to (2), (3), (4), and (5).

- (1) $\Rightarrow$ (2) This is clear.
- (3) $\Leftrightarrow$ (2) Let D_α be a bounded operator on $\mathscr{H}$; then $A = [A + \alpha(B+C)]D_\alpha$ if and only if $\alpha(B+C) = [A + \alpha(B+C)](I - D_\alpha)$. Now the equivalence of (2) and (3) follows from Theorem 1.7.
- (2) $\Rightarrow$ (1) If (2) holds, then (3) holds and hence (1) follows.
- (2) $\Rightarrow$ (4) If (2) holds, then by Theorem 1.7, there exists a bounded operator D_α such that $A = [A + \alpha(B+C)]D_\alpha$. Thus $A[A + \alpha(B+C) + \varepsilon]^{-1} = (D_\alpha)^*[A + \alpha(B+C)][A + \alpha(B+C)\varepsilon]^{-1}$, and hence $\|A[A+\alpha(B+C)+\varepsilon]^{-1}\| = \|(D_\alpha)^*[A+\alpha(B+C)][A+\alpha(B+C)+\varepsilon]^{-1}\| \leqslant \|(D_\alpha^*)\|$, because $[A + \alpha(B+C)][A + \alpha(B+C) + \varepsilon]^{-1} \leqslant I$. Thus (2) $\Rightarrow$ (4).
- (4) $\Rightarrow$ (2) Suppose (4) holds and let $x \in \mathscr{H}$. Then $\|Ax\| = \|A[A+\alpha(B+C)+\varepsilon]^{-1}[A+\alpha(B+C)+\varepsilon]x\| \leqslant M_1\|[A+\alpha(B+C)+\varepsilon]x\|$. It follows that $\|Ax\| \leqslant M_1\|[A + \alpha(B+C)]x\|$ and hence $A^2 \leqslant (M_1)^2 [A + \alpha(B+C) + \varepsilon]^2$. Therefore, by Theorem 1.7, $\text{Range}[A + \alpha(B+C)] \supseteq \text{Range}(A)$.
- (5) $\Rightarrow$ (3) The proof of this is similar to the proof of (2) $\Leftrightarrow$ (4).

Now the proof is complete.

2. Solution of $AX + BX = A$

In this section it is shown that the range inclusion on A and B guarantees a unique solution C satisfying conditions (a), (b), and (c) of Theorem 1.7 for the operator equation $AX + X^*B = A$. Also, it is shown that if $AB = BA$ then the operator equation $AX + XB = A$ has a unique positive solution C satisfying the above conditions.

THEOREM 2.1. *If* $\text{Range}(A + B) \supseteq \text{Range}(A)$, *then the operator equation* $AX + X^*B = A$ *has a unique solution* D *on* $\mathscr{H}$ *so that*

(a) $\|D\|^2 = \inf\{\mu / A^2 \leqslant \mu(A + B)^2\}$;

(b) $\text{Ker}(A) = \text{Ker}(D)$; *and*

(c) $\text{Range}(A + B)^- \supseteq \text{Range}(D)$.

Moreover D *is positive if and only if* $AB = BA$.

Proof. If $\text{Range}(A + B) \supseteq \text{Range}(A)$, then by Theorem 1.7, there exists a $\lambda \geqslant 0$ such that $A^2 \leqslant \lambda^2(A + B)^2$. Define a mapping C from $\text{Range}(A + B)$ to $\text{Range}(A)$ so that $C((A + B)x) = Ax$. Then C is well defined since

$$\|C(A + B)x\|^2 = \|Ax\|^2 \leqslant \lambda^2\|(A + B)x\|^2.$$

Hence C can be uniquely extended to $\mathrm{Range}(A+B)^{-}$, and if we define C on $\mathrm{Range}(A+B)^{\perp}$ to be 0, then $C(A+B)=A$. Moreover, $\langle C(A+B)x, (A+B)x\rangle = \langle Ax, (A+B)x\rangle = \langle x, (A^2+AB)x\rangle$ for all x. Thus C is positive if and only if $AB=BA$, bacause $A^2+AB \geqslant 0$ if and only if $AB=BA$.

Further consideration of the above proof shows us that (a) holds for the $D=C^*$. Also, (b) holds for this D because

$$\mathrm{Ker}(D) = \mathrm{Range}(D^*)^{\perp} = \mathrm{Range}(A)^{\perp} = \mathrm{Ker}(A)$$

and finally, (c) holds, because

$$\mathrm{Ker}(D^*) = \mathrm{Range}(D)^{\perp} \supseteq \mathrm{Range}(A+B)^{\perp},$$

which implies $\mathrm{Range}(A+B)^{-} \supseteq \mathrm{Range}(D)$. Now, we show that if E is an operator on $\mathscr{H}$ for which $A=(A+B)E$ and $\mathrm{Range}(A+B)^{-} \supseteq \mathrm{Range}(E)$, then $E=D$. If $A=(A+B)E$, then $E=D$ on $\mathrm{Range}(A+B)^{-}$. If $f \in \mathrm{Range}(A+B)^{\perp}$, then $f \in \mathrm{Range}(E)^{\perp} = \mathrm{Ker}(E)$ so that $Ef=0=Df$. Thus $E=D$.

Last, we show $A:B=BD$ and $B:A=(I-D)A$, which imply that $BD+DA=A$. By Theorem 1.6, the net $\{(B+\varepsilon):(A+\varepsilon)\}_{\varepsilon>0}$ converges in norm to $B:A$. Moreover, $(B+\varepsilon):(A+\varepsilon) = A+\varepsilon - A(A+B+2\varepsilon)^{-1}A - \varepsilon(A+B+2\varepsilon)^{-1}A - \varepsilon A(A+B+2\varepsilon)^{-1} - \varepsilon^2(A+B+2\varepsilon)^{-1}$.

Note that the last term goes to 0 in norm. Also, since

$$(A+B)(A+B+2\varepsilon)^{-1} \leqslant I$$

and

$$\begin{aligned}(A+B)^2(A+B+2\varepsilon)^{-1} - (A+B) &\\ = [(A+B)^2 - (A+B)(A+B+2\varepsilon)](A+B+2\varepsilon) &\\ = -2\varepsilon(A+B)(A+B+2\varepsilon)^{-1}, &\end{aligned}$$

one can conclude that $(A+B)(A+B2\varepsilon)^{-1}(A+B)-(A+B)$ is norm convergent to 0, and hence $(A+B)(A+B+2\varepsilon)^{-1}(A+B)$ converges to $A+B$. Therefore, $D^*(A+B)(A+B+2\varepsilon)^{-1}(A+B)D$ converges to $D^*(A+B)D$ and hence $A(A+B+2\varepsilon)^{-1}A$ converges in norm to AD. Furthermore, $\varepsilon(A+B)(A+B+2\varepsilon)^{-1}$ converges in norm to 0 and hence $\varepsilon A(A+B+2\varepsilon)^{-1}$ and $\varepsilon(A+B+2\varepsilon)^{-1}A$ both converge in norm to 0. Observation of all facts in this paragraph shows us that $(B+\varepsilon):(A+\varepsilon)$ converges in norm to $A-D^*(A+B)D=(A+B)D=(A+B)D(I-D)=A(I-D)$. Thus, $B:A=(B:A)^*=(I-D^*)A$. Similarly $A:B=BD$.

Last, since $B:A=A:B$, one can conclude that $BD=(I-D^*)A$. This shows that $BD+D^*A=A$ and the proof is complete.

The following result gives a sufficient condition for the solvability of $\alpha BX + X^*A = A$ for all real $\alpha > 0$.

LEMMA 2.1. *If $AB + BA \geqslant 0$, then for each real $\alpha > 0$, the equation $\alpha BX + X^*A = A$ has a unique bounded solution C_α satisfying the conditions* (a), (b), *and* (c) *in Theorem* 1.2.

Proof. If $AB + BA \geqslant 0$, then $\operatorname{Range}(A + \alpha B) \supseteq \operatorname{Range}(A)$ for all real $\alpha > 0$ by Theorem 1.5. The proof in the previous theorem implies that for each real $\alpha > 0$, there exists a unique bounded operator C_α satisfying the desired conditions such that $A : \alpha B = \alpha BC_\alpha$ and $\alpha B : A = A(I - C_\alpha)$. Moreover since $(\alpha B : A)^* = (\alpha B : A)$, one can conclude that $\alpha BC_\alpha = (I - C_\alpha^*)A$. This shows that $\alpha BC_\alpha + C_\alpha^* A = A$ and the proof is complete.

LEMMA 2.2. *If $AB + BA + A^2 \geqslant 0$, then $\alpha BX + X^*A = A$ has a unique bounded solution C_α satisfying the conditions* (a), (b), *and* (c) *in Theorem* 1.2 *for each real $\alpha \geqslant 1$.*

Proof. If $AB + BA + B^2 \geqslant 0$, then

$$\operatorname{Range}(A + \alpha B) \supseteq \operatorname{Range}(A)$$

and

$$\operatorname{Range}(A + \alpha B) \supseteq \operatorname{Range}(\alpha B) \qquad \text{for all real } \alpha \geqslant 1$$

by Theorem 1.4. The result now follows by Theorem 1.2.

THEOREM 2.2. *Let $AB = BA$; then the operator equation $\alpha BX + XA =$ has a unique bounded positive solution C_α satisfying the conditions* (a), (b), *and* (c) *in Theorem* 1.2 *for each real $\alpha \geqslant 1$.*

Proof. If $AB = BA$ then $\operatorname{Range}(A + B) \supseteq \operatorname{Range}(A)$ for all $\alpha > 0$ by Theorem 1.5. Now the result follows from Theorem 1.2.

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